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DIFFRACTION BY A TIPPED WEDGE

WITH APPLICATIONS TO BLUNT EDGES

by

SAM KARP

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Errata

Page 11	Eq. (3.2)	In denominator of integrand change " $\cos(\pi/n - \cos\|(\beta+\phi)/\gamma\|$ " to read " $\cos(\pi/x) - \cos[(\beta+\phi)/\gamma]$ "
Page 13,	5th line from bottom	In expression for $\|h_1(\theta,\frac{\pi}{2})\|$, change "$E_1^p$" to read "$H_1^p$"
Page 15,	8th line from bottom	Change " pages (16) and (17)" to read "pages (17-a) and (16) respectively"
Page 15,	4th line from bottom	Omit "for these angles"
Page 15,	4th line from bottom	Change "$h(\pi, \frac{\pi}{2})$" to read "$h_1(\pi, \frac{\pi}{2})$"
Page 17,		Change " $\|e(\theta, \frac{\pi}{2})\|$ " to read " $\|e_1(\theta, \frac{\pi}{2})\|$ " in caption and in designation of vertical ordinate.

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ABSTRACT

The problem of the diffraction of electromagnetic waves by a cylindrically tipped perfectly conducting wedge under plane wave or line source excitation is treated. Mathematically, this reduces to the problem of finding a solution of the two-dimensional time-reduced wave equation which either vanishes or has a vanishing normal derivative on the surface of the tipped wedge depending upon whether the electric vector or the magnetic vector is normal to the plane of incidence. Series representations of the solutions are obtained in all cases in terms of the eigenfunctions of the simple wedge problem (where the radius of the cylindrical tip is zero). The far field expressions for the case of the cylindrically tipped wedge under plane wave excitation are obtained for both polarizations. Graphs of the net scattered amplitude are given for the special case of normal incidence on a half-plane with a cylindrical tip of radius $a = k^{-1}$, k being the wave number.

1. Introduction

This report treats the two-dimensional problems arising from the diffraction of electromagnetic waves by a perfectly conducting cylindrically tipped wedge (cf. Figs. 1-(a) and 1-(b)) under plane-wave or line-source excitation. Mathematically this reduces to the problem of finding a solution of the two-dimensional time-reduced wave equation which either vanishes or has a vanishing normal derivative on the wedge depending upon whether the electric vector or the magnetic vector of the exciting field is normal to the plane of incidence. The solutions for the special case where the radius of the cylinder is zero are the well-known Sommerfeld solutions for the wedge (See [1] pp. 808-853 and [2] pp. 924-931). The solutions for the general case of the cylindrically tipped wedge are obtained by assuming each solution to be the sum of the corresponding Sommerfeld solution for the simple wedge and a perturbation arising from the cylindrical tip. After transforming to cylindrical coordinates, writing both the Sommerfeld solution and the corresponding perturbation fields in terms of a series of eigenfunctions in the simple wedge geometry and then equating coefficients of the former field to those of the latter on the surface of the cylinder, we obtain the eigenfunction expansion of the perturbation field and hence of the total field. Thus we have a modified Sommerfeld problem for which we can obtain a solution which holds for all angles of the cylindrically tipped wedge, and in particular for the case of the cylindrically tipped half-plane. The eigenfunction series forms of the total fields for plane-wave and line-source excitation are given for both polarizations and for any wedge angle, but computations of the far field patterns are confined to the case of the cylindrically tipped half-plane.

Graphs are given of the net scattered amplitudes in regions other than the transition regions[*] for the case where the incidence is normal to the half-plane and the radius a of the cylinder is such that $ka = 1$.

One might conjecture that for suitably chosen values of ka the diffracted field resulting from a cylindrically tipped half-plane could be employed to describe the field resulting from diffraction by bodies not well represented by the perfectly thin and sharp half-plane of Sommerfeld's theory. The author brought this possibility to the attention of Mr. R.V. Row[**], who was engaged at the time in making microwave measurements on the effect of thickness of the half-plane on the field in the neighborhood of the shadow line of the normally incident plane wave. Row made measurements on the electric field due to a half-plane which is thin compared to the wavelength and which has a cylindrical tip of diameter D. He also made measurements on the field due to diffraction by a screen of thickness D and compared the results for the typical cases $D = \frac{\lambda}{4}$ and $D = \frac{\lambda}{8}$ with the electric field which he computed using the present theory for the half-plane with a tip of diameter D. His results were published recently as part of a paper on microwave diffraction measurements on a parallel plate region [7]; there appears to be some agreement with the theory for the region of space he considered.

It should be mentioned that the case of a 180° wedge angle, i.e., the case of an infinite plane surmounted by a ridge of semicircular cross-section has been treated by Rayleigh [3], and subsequently and independently by Tai [4] and Twersky [5]. Moullin [6] noted that in the case of wedge angles of the form π/n (the angle Ψ in Fig. 1) the well-known solusolution by images for integral values of n can be extended to the case where there is a cylindrical tip. He correctly assumes the solution to hold also for non-integral n, for instance for $n = \frac{1}{2}$ which corresponds to the angle 2π, i.e., to the half-plane. The present solution, obtained independently, may be regarded as a generalization for a more general wedge angle using only elementary means.

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[*] The transition regions are angular regions about the shadow lines of the incident and reflected waves occurring in this case at $\Theta = 270^\circ$ and $\Theta = 90^\circ$ respectively (cf. Fig. 1-(a) and 1-(b) for $\Psi = 0$).

[**] In response to a query concerning a certain work involving the diffraction of a thin plane of rectangular cross-section.

2. Formulation and Solution of the Problem

2.1 Statement of the Problem

Let W_a be a perfectly conducting solid wedge with a cylindrical tip of radius a. Its surface, S_a, is composed of two plane surfaces, say L_a^1 and L_a^2, and the cylindrical tip C_a. In terms of the cylindrical coordinates

$$\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= z, \end{aligned} \tag{2.1}$$

S_a may be characterized analytically by the following relations:

$$Sa = L_a^1 + C_a + L_a^2 ;$$

where for

$$L_a^1 , \ \theta = 0, \ \rho > a, \ -\infty < z < \infty,$$

for

$$C_a, \ \rho = a \quad 0 \leq \theta \leq 2\pi - \Psi, \quad -\infty < z < \infty, \tag{2.2}$$

and for

$$L_a^2, \ \theta = 2\pi - \Psi, \ \rho > a, \ -\infty < z < \infty.$$

The wedge angle Ψ may have any value in the range $o \leq \Psi \leq 2\pi$. In figures 1-(a) and 1-(b) we have depicted the X- Y cross-section of this wedge. On setting $a = 0$ in Eq. (2.2) it is clear that S_a reduces to the

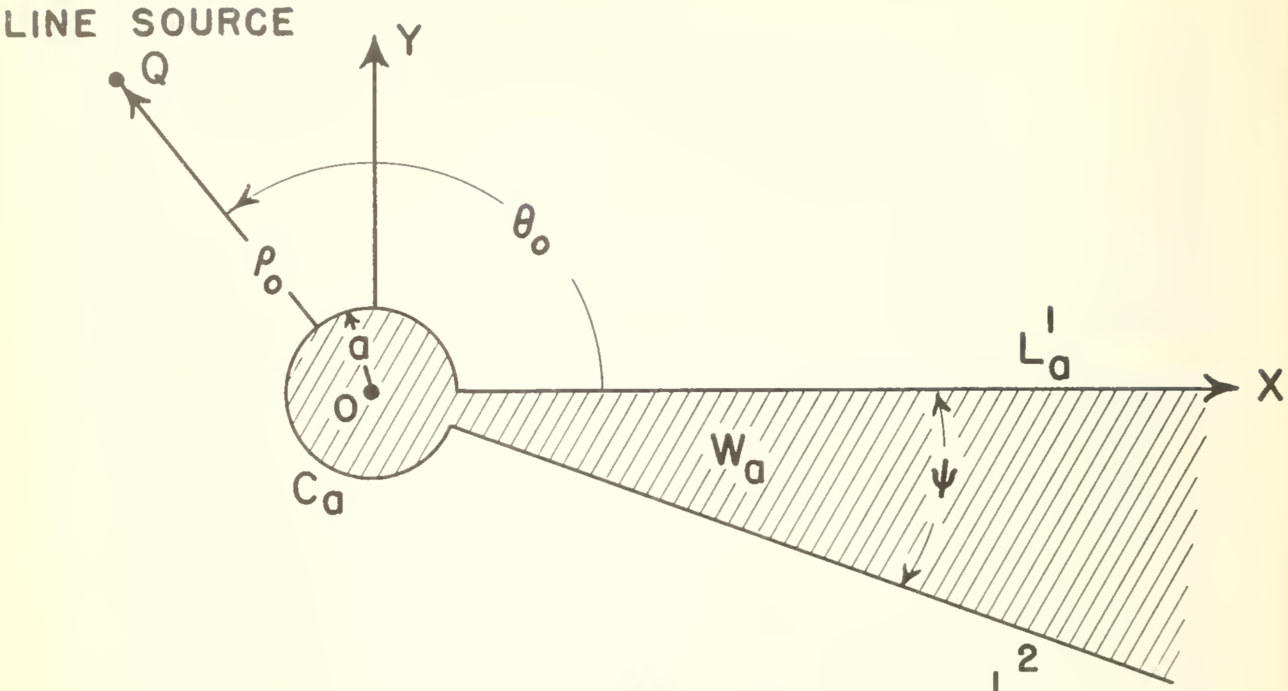

LINE SOURCE
Q
Y
θ_o
ρ_o
a
O
L_a^1
X
W_a
ψ
C_a
L_a^2

Fig. 1(a)

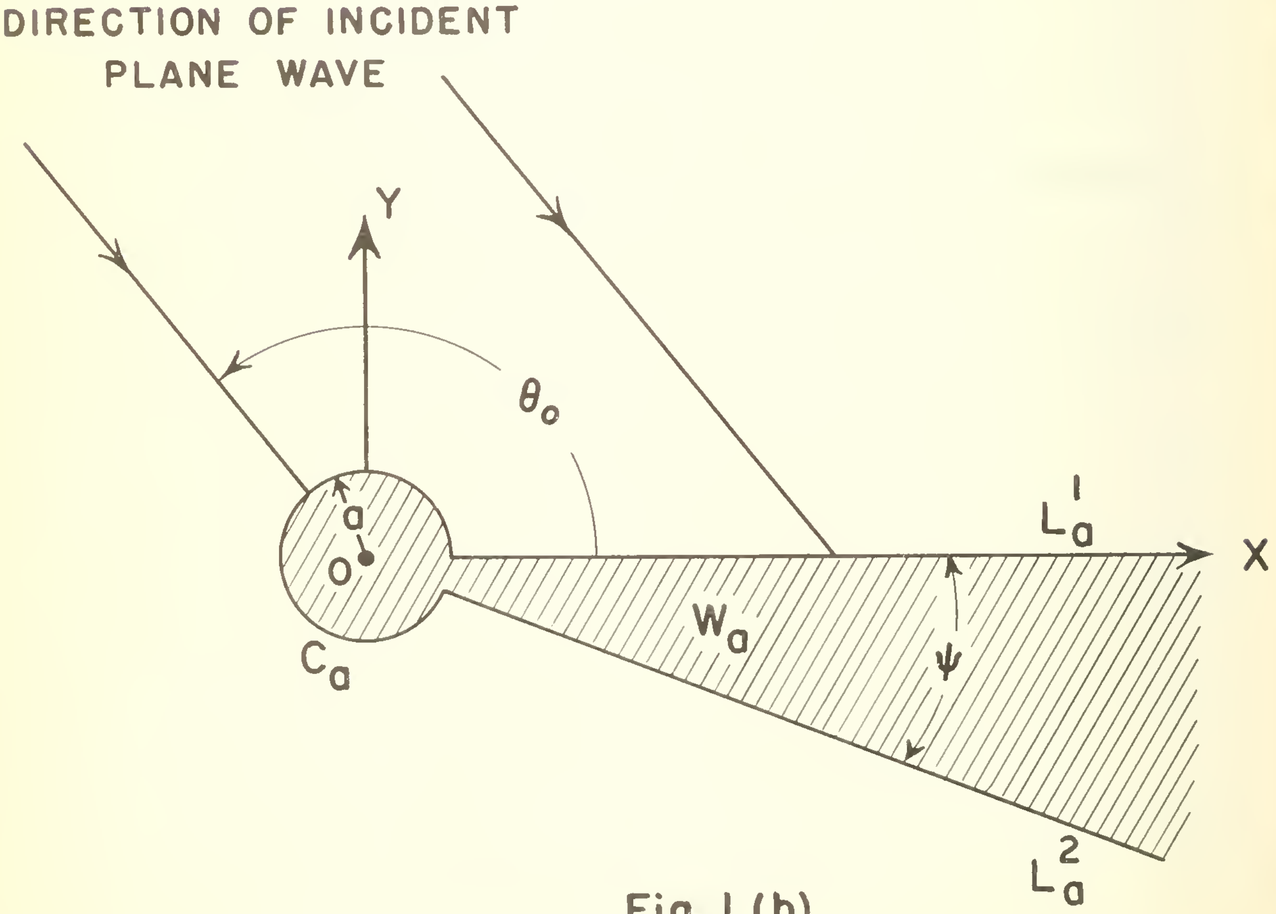

DIRECTION OF INCIDENT
PLANE WAVE
Y
θ_o
a
O
L_a^1
X
W_a
ψ
C_a
L_a^2

Fig. 1 (b)

surface $S_o = L_o^1 + C_o + L_o^2$, i.e., to a simple wedge surface whose leading edge is parallel to the Z-axis.

Since we shall ultimately excite the wedge by a plane wave or cylindrical wave which has a time-dependence of $e^{i\omega t}$ and no z-dependence, we may assume that all vectors in the following considerations have the time-dependence $e^{i\omega t}$ and are independent of z. Letting $\bar{E}(x,y)e^{i\omega t}$ and $\bar{H}(x,y)e^{i\omega t}$ be the total electric and magnetic field vectors respectively, we obtain for Maxwell's equations in free space

$$\begin{aligned} \nabla \times \bar{E} + \frac{i\omega}{c}\bar{H} &= 0, \\ \nabla \times \bar{H} - \frac{i\omega}{c}\bar{E} &= 0. \end{aligned} \tag{2.3}$$

Furthermore, since the wedge is a perfect conductor the boundary conditions become

$$\bar{E} \times \bar{n} = \bar{H} \times \bar{n} = 0 \text{ on } S_a, \tag{2.4}$$

where $\bar{n}$ is the unit normal to S_a. Now any solution of Eq. (2.3) may be written in the form

$$\begin{aligned} \bar{E} &= E_z\,\bar{z}_o + \bar{E}^n \\ \bar{H} &= H_z\,\bar{z}_o + \bar{H}^n , \end{aligned} \tag{2.5}$$

where the vectors $E_z\,\bar{z}_o(H_z\,\bar{z}_o)$ and $\bar{E}^n$ $(\bar{H}^n)$ are parallel and perpendicular to the Z-axis respectively ($\bar{z}_o$ is a unit vector parallel to the z-axis). If in addition $\bar{E}$ and $\bar{H}$ satisfy the boundary conditions (2.4) then, employing the usual vector methods one can easily verify that $\bar{E}$, $\bar{H}$ may be obtained, in accordance with prescription of Eq. (2.5), by superposing the solutions $E_z\,\bar{z}_o$, $\bar{H}^n$ and $\bar{E}^n$, $H_z\,\bar{z}_o$ of the following two boundary-value problems respectively:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k^2 E_z = 0; \qquad k^2 = \omega^2 \mu\varepsilon$$

(2.6)

$$E_z = 0 \text{ on } S_a,$$

$$\bar{H}^n = -\frac{1}{i\omega\varepsilon}\,\bar{z}_o \times \nabla E_z$$

and

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k^2 H_z = 0;$$

(2.7)

$$\frac{\partial H_z}{\partial n} = 0 \text{ on } S_a,$$

$$\bar{E}^n = \frac{1}{i\omega\mu}\,\bar{z}_o \times \nabla H_z \,.$$

It is clear therefore that the solutions E_z and H_z of the first two equations of Eq. (2.6) and Eq. (2.7) respectively determine the entire electromagnetic field.

In what follows we shall assume either plane-wave or line-source excitation. In the case of plane-wave excitation we suppose E_z and H_z to be of the form

(2.8)

$$E_z = e^{ik\rho\cos(\theta-\theta_o)} + \text{scattered wave}$$

$$H_z = e^{ik\rho\cos(\theta-\theta_o)} + \text{scattered wave},$$

where θ_o is the angular direction of the ray from the origin to the source at infinity as measured from the positive X-axis, [cf. Fig.1-(a)]. In the case of line-source excitation we shall assume

(2.9)

$$E_z = H_o^{(2)}\left(k\sqrt{\rho^2+\rho_o^2-2\rho_o\rho\cos(\theta-\theta_o)}\,\right) + \text{scattered wave}$$

$$H_z = H_o^{(2)}\left(k\sqrt{\rho^2+\rho_o^2-2\rho_o\rho\cos(\theta-\theta_o)}\,\right) + \text{scattered wave}.$$

where (ρ_o,θ_o) $\rho_o > a$, are the polar coordinates of the line source [cf.Fig.1-(b)].

In all cases we shall require that the scattered waves satisfy the radiation condition at infinity.

In the following sections we shall drop the subscript in E_z (H_z) and write E_1 (H_1) or E_2 (H_2) depending upon whether we are dealing with the case of plane-wave or line-source excitations.

2.2 Eigenfunction Series Solution for Plane-Wave and Line-Source Excitation of W_a

To obtain eigenseries solutions for the various problems posed in the previous section we decomposes each solution into the sum of two parts:

(a) the solution of the simple W_o - wedge problem for the corresponding boundary condition, and

(b) the perturbation of the field caused by the cylindrical tip. Thus we write

$$(2.10) \qquad \begin{aligned} E_j &= E_j^o + E_j^p \qquad && j = 1,2 \\ H_j &= H_j^o = H_j^p \qquad && j = 1,2, \end{aligned}$$

where E_j^o , H_j^o are the solutions for the simple wedge problem and where E_j^p, H_j^p are the perturbations arising from the cylindrical tip. Now the solutions E_j^o and H_j^o of the two-dimensional time-reduced wave equation which satisfy the prescribed boundary conditions [cf.Eqs. (2.6) and (2.7)] may be obtained from the well-known Sommerfeld solutions for the wedge [cf.(5), chap. XX]. For the case of plane-wave excitation [cf. Eq. (28)] the solutions, in eigenseries form, are

$$(2.11) \qquad \begin{aligned} E_1^o(r,\theta) &= \frac{4}{\gamma} \sum_{n=1}^{\infty} e^{i \frac{n}{\gamma} \frac{\pi}{2}} J_{\frac{n}{\gamma}}(k\rho) \sin \frac{n}{\gamma}\, \theta_o \sin \frac{n\theta}{\gamma}, \qquad && 0 \leq \theta \leq 2\pi - \Psi, \\ H_1^o(r,\theta) &= \frac{2}{\gamma} J_o(k\rho) + \frac{4}{\gamma} \sum_{n=1}^{\infty} e^{i \frac{n}{\gamma} \frac{\pi}{2}} \cos \frac{n}{\gamma}\, \theta_o \cos \frac{n\theta}{\gamma}, \qquad && 0 \leq \theta \leq 2\pi - \Psi, \end{aligned}$$

where

$$\gamma = \frac{2\pi - \Psi}{\pi}, \quad 0 \leq \Psi < 2\pi. \tag{2.12}$$

For the case of line-source excitation [cf.Eq.(2.9)] the solutions for $0 \leq \theta \leq 2\pi - \Psi$, in eigenseries form, are

$$\begin{aligned}
E_2^o &= \frac{4}{\gamma} \sum_{n=1}^{\infty} H^{(2)}_{\frac{n}{\gamma}}(k\rho_o) J_{\frac{n}{\gamma}}(k\rho) \sin \frac{n}{\gamma}\theta_o \sin \frac{n}{\gamma}\theta, && 0 \leq \rho < \rho_o, \\
E_2^o &= \frac{4}{\gamma} \sum_{n=1}^{\infty} J_{\frac{n}{\gamma}}(k\rho_o) H^{(2)}_{\frac{n}{\gamma}}(k\rho) \sin \frac{n}{\gamma}\theta_o \sin \frac{n}{\gamma}\theta, && \rho > \rho_o; \\
H_2^o &= \frac{2}{\gamma} H_o^{(2)}(k\rho_o) J_o(k\rho) + \frac{4}{\gamma} \sum_{n=1}^{\infty} H^{(2)}_{\frac{n}{\gamma}}(k\rho_o) J_{\frac{n}{\gamma}}(k\rho) \cos\frac{n}{\gamma}\theta_o \cos\frac{n}{\gamma}\theta, && \rho < \rho_o \\
H_2^o &= \frac{2}{\gamma} J_o(k\rho_o) H_o^{(2)}(k\rho) + \frac{4}{\gamma} \sum_{n=1}^{\infty} J_{\frac{n}{\gamma}}(k\rho_o) H^{(2)}_{\frac{n}{\gamma}}(k\rho) \cos\frac{n}{\gamma}\theta_o \cos\frac{n}{\gamma}\theta, && \rho > \rho_o,
\end{aligned} \tag{2.13}$$

where γ is defined in Eq. (2.12).

Equations (2.11) and (2.13) may be written in the form

$$\begin{aligned}
E_1^o(\rho,\theta) &= \sum_{n=1}^{\infty} e_n(\rho) \sin \frac{n}{\gamma}\theta_o \sin \frac{n\theta}{\gamma}, && 0 \leq \theta \leq 2\pi - \Psi,\ 0 \leq \rho < \infty \\
H_1^o(\rho,\theta) &= \sum_{n=o}^{\infty} h_n(\rho) \cos \frac{n}{\gamma}\theta_o \cos \frac{n\theta}{\gamma}, && 0 \leq \theta \leq 2\pi - \Psi,\ 0 \leq \rho < \infty
\end{aligned} \tag{2.14}$$

where $e_n(\rho)$ and $h_n(\rho)$ are the appropriate functions of ρ and depend upon the nature of the excitation. Although we shall confine ourselves to the case of plane and line-source excitation, it should be mentioned that the derivation of the perturbed field, which follows below is valid even when e_n and h_n are due to a general cylindrical source having the appropriate polarization, or to a combination of such sources, provided only that $e_n(\rho)$ and $h_n(\rho)$ are the functional forms which are relevant for $a < \rho < R$, where R is the distance from the origin to the nearest source.

It is now an easy matter to find the eigenseries expansions for the perturbation fields E_j^p and H_j^p, $j = 1,2$. Let us write

$$(2.15)\qquad \begin{aligned} E_1^p &= \sum_{n=1}^{\infty} a_n H^{(2)}_{\frac{n}{\gamma}}(k\rho)\sin\frac{n\Theta_o}{\gamma}\sin\frac{n\Theta}{\gamma}, & \rho \geq a,\ 0 \leq \Theta \leq 2\pi-\Psi, \\ H_1^p &= \sum_{n=o}^{\infty} b_n H_{\frac{n}{\gamma}}(k\rho)\cos\frac{n\Theta_o}{\gamma}\cos\frac{n\Theta}{\gamma}, & \rho \geq a,\ 0 \leq \Theta \leq 2\pi-\Psi \end{aligned}$$

where $H_m^{(2)}(z)$ is the Hankel function of the second kind and is of order m. Since we have assumed the time dependence $e^{i\omega t}$ and since

$$H_m^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(z - \frac{2m+1}{2}\right)\right],\ |z| \to \infty,$$

each term in the series expression for E_1^p (H_1^p) represents an outgoing wave at infinity. It is clear now that $E_1^p(\frac{\partial H_1^p}{\partial \Theta})$ vanishes on the planar portions of S_a, i.e., on the surfaces L_a^1 and L_a^2. In order that the total field E_1 $(\frac{\partial H_1}{\partial \Theta})$ vanish on the cylindrical tip C_a, i.e., on the surface determined by

$$(2.16)\qquad \rho = a,\ 0 \leq \Theta \leq 2\pi-\Psi,$$

one need only equate coefficients in like terms of E_1^p $(\frac{\partial H_1^p}{\partial r})$ and $E_1^o(\frac{\partial H_1^o}{\partial r})$ for $\rho = a$. The results are

$$(2.17)\qquad \begin{aligned} a_n &= -\frac{e_n(a)}{H^{(2)}_{\frac{n}{\gamma}}(ka)}, & n = 1,2,\cdots \\ b_n &= -h_n^{1\prime}(a) \Big/ H^{(2)\prime}_{\frac{n}{\gamma}}(z)\Big|_{z\,=\,ka}, & n = o,1,2,\cdots\ ; \end{aligned}$$

The expressions for E_1^p and H_1^p therefore become

$$E_1^p = -\frac{4}{\gamma}\sum_{n=1}^{\infty} e^{i\frac{n}{2}\frac{\pi}{\gamma}}\left(\frac{J_{\frac{n}{\gamma}}(ka)}{H^{(2)}_{\frac{n}{\gamma}}(ka)}\right)H^{(2)}_{\frac{n}{\gamma}}(k\rho)\sin\frac{n}{\gamma}\theta_0\sin\frac{n}{\gamma}\theta, \qquad \rho \geq a$$

(2.18)

$$H_1^p = -\frac{2}{\gamma}\left(\frac{J_0'(z)}{H_0^{(2)\prime}(z)}\bigg|_{z = ka}\right)H_0^{(2)}(k\rho) - \frac{4}{\gamma}\sum_{n=1}^{\infty} e^{i\frac{n}{2}\frac{\pi}{\gamma}}\left(\frac{J'_{\frac{n}{\gamma}}(z)}{H^{(2)\prime}_{\frac{n}{\gamma}}(z)}\bigg|_{z = ka}\right)$$

$$H^{(2)}_{\frac{n}{\gamma}}(k\rho)\cos\frac{n\theta_0}{\gamma}\cos\frac{n\theta}{\gamma}, \qquad \rho \geq a.$$

Since the general terms in the series expressions for E_1^p (H_1^p) satisfy the wave equation it is clear that the total fields of Eq. (2.10) constitute solutions of this equation which satisfy the prescribed boundary conditions.

For the case of line-source excitation we have, corresponding to Eq. (2.18), the result

$$E_2^p = -\frac{4}{\gamma}\sum_{n=0}^{\infty} H^{(2)}_{\frac{n}{\gamma}}(k\rho_0)\left(\frac{J_{\frac{n}{\gamma}}(ka)}{H^{(2)}_{\frac{n}{\gamma}}(ka)}\right)H^{(2)}_{\frac{n}{\gamma}}(k\rho)\sin\frac{n}{\gamma}\theta_0\sin\frac{n}{\gamma}\theta, \qquad \rho \geq a$$

(2.19)

$$H_2^p = -\frac{2}{\gamma}H_0^{(2)}(k\rho_0)\left(\frac{J_0'(z)}{H_0^{(2)\prime}(z)}\bigg|_{z=ka}\right) - \frac{4}{\gamma}\sum_{n=1}^{\infty} H^{(2)}(k\rho_0)\left(\frac{J'_{\frac{n}{\gamma}}(z)}{H^{(2)}_{\frac{n}{\gamma}}(z)}\bigg|_{z=ka}\right)$$

$$H^{(2)}_{\frac{n}{\gamma}}(k\rho)\cos\frac{n\theta_0}{\gamma}\cos\frac{n\theta}{\gamma}, \qquad \rho \geq a.$$

In our investigations of the far fields we shall be particularly interested in the case where the wedge reduces to a half-plane with a cylindrical tip. If we set $\Psi = 0$, the parameter $\gamma = 2\pi - \Psi/\pi$ becomes $\gamma = 2$. Inserting this value for γ in Eq. (2.18) we have

$$E_1^p = -2 \sum e^{i\frac{n\pi}{4}} \frac{J_{\frac{n}{2}}(ka)}{H^2_{\frac{n}{2}}(ka)} \; H^{(2)}_{\frac{n}{2}}(k\rho)\sin\frac{n\theta_o}{2}\sin\frac{n\theta}{2}\,, \qquad \rho \geq a$$

(2.20)

$$H_1^p = -\left(\frac{J_o'(z)}{H_o^{(2)\prime}(z)}\Bigg|_{z=ka}\right) H_o^{(2)}(k\rho) - 2\sum_{n=1}^{\infty} e^{i\frac{n\pi}{4}}\left(\frac{J'_{\frac{n}{2}}(z)}{H^{(2)\prime}_{\frac{n}{2}}(z)}\Bigg|_{z=ka}\right)$$

$$H^{(2)}_{\frac{n}{2}}(k\rho)\cos\frac{n\theta_o}{2}\cos\frac{n\theta}{2}\,, \qquad \rho \geq a.$$

3. The Far Fields for a Plane Wave Incident on a Cylindrically Tipped Wedge

3.1 The Far Field Expressions of E_1^o and H_1^o

In this section and the following one we shall determine the far field behavior of the cylindrically tipped wedge under plane wave excitation. The procedure employed may also be used in the case of the wedge W_a under line-source excitation; however we shall not consider this case here.

The computation of the far fields of E_1 and H_1 reduces in principle to the computation of the far fields of E_1^p and H_1^p since the desired expressions for E_1^o and H_1^o have been repeatedly given and discussed in the literature. In this section we collect those far field formulas for E_1^o and H_1^o which are useful for our purposes.

For any angular direction θ_o, $0 < \theta_o < 2\pi-\Psi$ of the source at infinity, the far fields of E_1 and H_1 may be written in the form

$$E_1 = v_B(k\rho,\theta-\theta_o) - v_B(k\rho,\theta+\theta_o) + \text{plane waves}$$

(3.1)

$$H_1 = v_B(k\rho,\theta-\theta_o) + v_B(k\rho,\theta+\theta_o) + \text{plane waves},$$

where (cf. [2] p.926)

$$(3.2) \qquad v_B(k\rho,\emptyset) = \frac{1}{2\pi i}\,\frac{\sin\pi/\gamma}{\gamma}\int_{-i\infty-\varepsilon}^{i\infty+\varepsilon}\frac{e^{-ik\rho\cos\beta}\,d\beta}{\cos(\pi/n-\cos|(\beta+\emptyset)/\gamma|}\,,^{*}$$

the parameter ε being any angle between zero and π. The plane wave portions of Eq. (3.1) consist of the incident wave and the reflected waves which appear as expected from considerations of geometrical optics. Thus, for example, when $\pi-\Psi < \theta_o < \pi$, there appear two reflected waves, one from the upper surface and one from the lower surface of the wedge. Employing the method of steepest descent, one then has

$$(3.3) \qquad v_B(k\rho,\emptyset) \sim \frac{e^{-i(k\rho+\frac{\pi}{4})}}{\sqrt{2\pi k\rho}}\;\frac{\sin\pi/\gamma}{\gamma(\cos\pi/\gamma-\cos\emptyset/\gamma)}$$

whenever

$$(3.4) \qquad k\rho\left(\cos\frac{\pi}{\gamma}-\cos\frac{\emptyset}{\gamma}\right)^2 >> 1,$$

i.e., whenever $\emptyset$ is not in the neighborhood of π. Substituting the relation (3.3) into Eq. (3.1) we then have

$$(3.4) \qquad \begin{aligned} E_1 &\sim \frac{e^{-i(k\rho+\frac{\pi}{4})}}{\sqrt{2\pi k\rho}}\;\frac{\sin\pi/\gamma}{\gamma}\left\{\frac{1}{\cos\frac{\pi}{\gamma}-\cos\frac{\theta-\theta_o}{\gamma}}-\frac{1}{\cos\frac{\pi}{\gamma}-\cos\frac{\theta+\theta_o}{\gamma}}\right\}+\text{plane waves} \\ H_1 &\sim \frac{e^{-i(k\rho+\frac{\pi}{4})}}{\sqrt{2\pi k\rho}}\;\frac{\sin\pi/\gamma}{\gamma}\left\{\frac{1}{\cos\frac{\pi}{\gamma}-\cos(\frac{\theta-\theta_o}{\gamma})}+\frac{1}{\cos\frac{\pi}{\gamma}-\cos(\frac{\theta+\theta_o}{\gamma})}\right\}+\text{plane waves.} \end{aligned}$$

In particular for the case of a half-plane (i.e., when $\gamma = 2$) we have

---

* With v_B in this form one can, by shifting contours properly, arrive at the well-known Sommerfeld integral solution for E_1 and H_1. From this form of the solution, following the method in [1], pp.838-839, one obtains easily the Bessel function expansion of Eq. (2.11).

$$(3.5)\qquad \begin{aligned} E_1 &\sim -\frac{e^{-i(k\rho+\frac{\pi}{4})}}{2\sqrt{2\pi k\rho}}\left\{\sec\left(\frac{\theta-\theta_o}{2}\right)-\sec\left(\frac{\theta+\theta_o}{2}\right)\right\} + \text{plane waves} \\ H_1 &\sim -\frac{e^{-i(k\rho+\frac{\pi}{4})}}{2\sqrt{2\pi k\rho}}\left\{\sec\left(\frac{\theta-\theta_o}{2}\right)+\sec\left(\frac{\theta+\theta_o}{2}\right)\right\} + \text{plane waves.} \end{aligned}$$

3.2 The Far-Field Expressions for $E_1^p(r,\theta)$ and $H_1^p(r,\theta)$

In this section we write the far field expressions for $E_1^p(r,\theta)$ and $H_1^p(r,\theta)$ in a form suitable for computation.

First of all for $|z| \geq 1$ and $|z| >> m$ we have

$$(3.6)\qquad H_m^{(2)}(z)\sim\sqrt{\frac{2}{\pi z}}\ \exp\left[-i\left(z-\frac{2m+1}{4}\pi\right)\right],\ z\to\infty.$$

Substituting into both expressions of Eqs. (1.18) and rearranging terms we get

$$(3.7)\qquad \begin{aligned} E_1^p &\sim -\frac{1}{\gamma\sqrt{2\pi k\rho}}\exp\left[-i\left(k\rho+\frac{\pi}{4}\right)\right]\left\{8i\sum_{n=1}^{\infty} e^{i\frac{n\pi}{\gamma}}\ \frac{J_{\frac{n}{\gamma}}(ka)}{H^{2}_{\frac{n}{\gamma}}(ka)}\ \sin\frac{n\theta_o}{\gamma}\ \sin\frac{n\theta}{\gamma}\right\}, \qquad \rho \geq a \\ H_1^p &\sim -\frac{1}{\gamma\sqrt{2\pi k\rho}}\exp\left[-i\left(k\rho+\frac{\pi}{4}\right)\right]\left\{4i\frac{J_o'(z)}{H_o^{(2)\prime}(z)}\bigg|_{z=ka} + 8i\sum_{n=1}^{\infty} e^{i\frac{n\pi}{\gamma}}\ \frac{J'_{\frac{n}{\gamma}}(z)}{H^{(2)\prime}_{\frac{n}{\gamma}}(z)}\bigg|_{z=ka} \right. \\ &\qquad\qquad \left. \cos\frac{n\theta_o}{\gamma}\ \cos\frac{n\theta}{\gamma}\right\}, \qquad p \geq a \end{aligned}$$

at least when ka is sufficiently small to insure the negligibility of the terms in which n is large.

In particular for the case of the cylindrically tipped half-plane we have, on setting $\gamma = 2$ in Eq. (3.7),

$$E_1^p \sim -\frac{e^{-i(k\rho+\frac{\pi}{4})}}{2\sqrt{2\pi k\rho}}\left\{8i\left(\sum_{n=1}^{\infty} i^n \frac{J_{\frac{n}{2}}(ka)}{H^{(2)}_{\frac{n}{\gamma}}(ka)} \sin\frac{n\theta_o}{2}\sin\frac{n\theta}{2}\right)\right\}, \qquad \rho \geq a$$

$$(3.8) \qquad H_1^p \sim -\frac{e^{-i(k\rho+\frac{\pi}{4})}}{2\sqrt{2\pi k\rho}}\left\{4i\frac{J_o'(z)}{H_o^{(2)\,'}(z)}\bigg|_{z=ka} + 8i\sum_{n=1}^{\infty} i^n \frac{J'_{\frac{n}{2}}(z)}{H^{(2)}_{\frac{n}{2}}(z)}\bigg|_{z=ka} \cos\frac{n\theta_o}{2}\cos\frac{n\theta}{2}\right\}, \qquad \rho \geq a.$$

In what follows, since we shall confine our computations to the case of the cylindrically tipped half-plane, we shall only consider Eqs. (3.5) and (3.8). From these equations one sees that the far fields of E_1 and H_1 are of the form

$$(3.9) \qquad \begin{aligned} E_1 &\sim A(\rho)e_1(\theta,\theta_o) + \text{plane waves} \\ H_1 &\sim A(\rho)h_1(\theta,\theta_o) + \text{plane waves}, \end{aligned}$$

where

$$(3.10) \qquad A(\rho) = -\frac{e^{-i(k\rho+\frac{\pi}{4})}}{2\sqrt{2\pi k\rho}} .$$

Our aim is to compute $|e_1(\theta,\theta_o)|$ and $|h_1(\theta,\theta_o)|$ for $\theta_o = \frac{\pi}{2}$. To this end we must first compute the quantities $\mathrm{Re}(E_1^p/A(\rho))$, $\mathrm{Re}(H_1^p/A(\rho))$, $\mathrm{Im}(E_1^p/A(\rho))$ and $\mathrm{Im}(H_1^p/A(\rho))$. With these computations performed, $|e_1(\theta,\frac{\pi}{2})|$ and $|h_1(\theta,\frac{\pi}{2})|$ may be expressed as

$$(3.11) \qquad \begin{aligned} |e_1(\theta,\tfrac{\pi}{2})| &= \sqrt{\left[\mathrm{Im}\left(\frac{E_1^p}{A(\rho)}\right)\right]^2 + \left[\mathrm{Re}\left(\frac{E_1^p}{A(\rho)}\right) + \sec\left(\frac{\theta-\frac{\pi}{2}}{2}\right) - \sec\left(\frac{\theta+\frac{\pi}{2}}{2}\right)\right]^2} \\ |h_1(\theta,\tfrac{\pi}{2})| &= \sqrt{\left[\mathrm{Im}\left(\frac{H_1^p}{A(\rho)}\right)\right]^2 + \left[\mathrm{Re}\left(\frac{E_1^p}{A(\rho)}\right) + \sec\left(\frac{\theta-\frac{\pi}{2}}{2}\right) + \sec\left(\frac{\theta+\frac{\pi}{2}}{2}\right)\right]^2} . \end{aligned}$$

To express E_1^p (H_1^p) in its real and imaginary parts we proceed as follows: We write

$$(3.12) \qquad \begin{aligned} H^{(2)}_{\frac{n}{2}}(q) &= J_{\frac{n}{2}}(q) - i\,N_{\frac{n}{2}}(q) \\ H^{(2)\,'}_{\frac{n}{2}}(q) &= J'_{\frac{n}{2}}(q) - i\,N'_{\frac{n}{2}}(q). \end{aligned}$$

Now

$$\arctan q = \frac{\pi}{2} - \arctan \frac{1}{q} ;$$

and hence on setting

(3.13)
$$\delta_{\frac{n}{2}}(q) = \arctan \left[- \frac{J_{\frac{n}{2}}(q)}{N_{\frac{n}{2}}(q)} \right]$$
$$\delta'_{\frac{n}{2}}(q) = \arctan \left[- \frac{J'_{\frac{n}{2}}(q)}{N'_{\frac{n}{2}}(q)} \right] ,$$

we get

(3.14)
$$\arctan \left[- \frac{N_{\frac{n}{2}}(q)}{J_{\frac{n}{2}}(q)} \right] = \frac{\pi}{2} - \delta_{\frac{n}{2}}(q) \equiv \nu_{\frac{n}{2}}(q), \; n = 1,2,$$
$$\arctan \left[- \frac{N'_{\frac{n}{2}}(q)}{J'_{\frac{n}{2}}(q)} \right] = \frac{\pi}{2} - \delta'_{\frac{n}{2}}(q) \equiv \nu'_{\frac{n}{2}}(q), \; n = 1,2,\cdots .$$

It follows that

(3.15)
$$\frac{J_{\frac{n}{2}}(q)}{H^{(2)}_{\frac{n}{2}}(q)} = \cos \nu_{\frac{n}{2}}(q) \left[\cos \nu_{\frac{n}{2}}(q) - i \sin \nu_{\frac{n}{2}}(q)\right]$$
$$= \sin \delta_{\frac{n}{2}}(q) \left[\sin \delta_{\frac{n}{2}}(q) - i \cos \delta_{\frac{n}{2}}(q)\right]$$
$$\frac{J'_{\frac{n}{2}}(q)}{H^{(2)\prime}_{\frac{n}{2}}(q)} = \sin \delta'_{\frac{n}{2}}(q) \left[\sin \delta'_{\frac{n}{2}}(q) - i \cos \delta'_{\frac{n}{2}}(q)\right].$$

Thus on setting

(3.16) $$ka = q,$$

we have for $0 \leq \theta \leq 2\pi$,

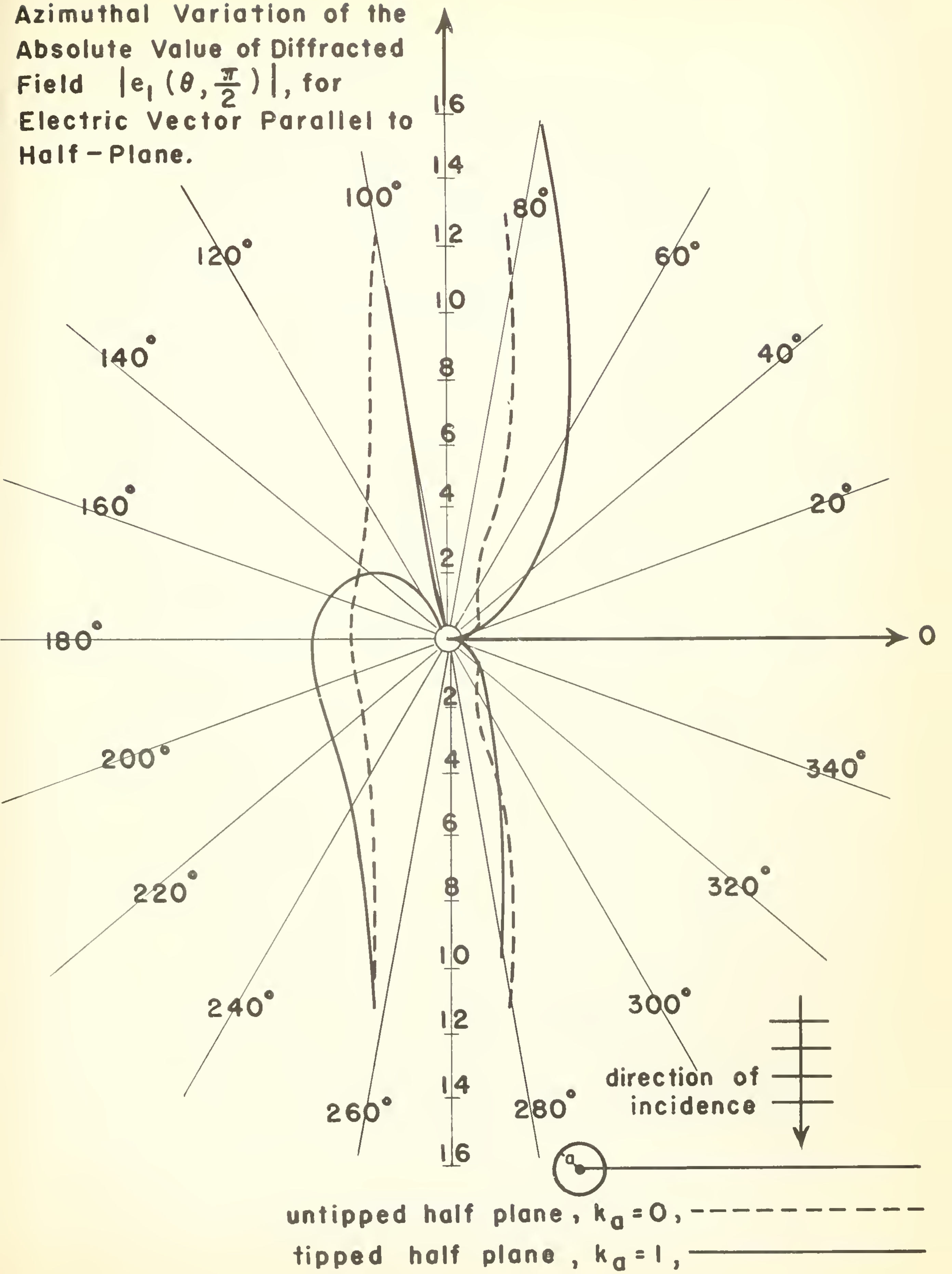
$$
\begin{aligned}
\operatorname{Re}\left(\frac{E_1^p}{A(\rho)}\right) &= 8\Bigg\{\sum_{n=1}^{\infty}(-1)^n\sin^2\delta_{\frac{2n-1}{2}}(q)\sin\frac{2n-1}{2}\,\theta_o\sin\frac{2n-1}{2}\,\theta \\
&\qquad + \sum_{n=1}^{\infty}(-1)^n\sin\delta_n(q)\cos\delta_n(q)\sin n\theta_o\sin n\theta\Bigg\} \\
\operatorname{Im}\left(\frac{E_1^p}{A(\rho)}\right) &= 8\Bigg\{\sum_{n=1}^{\infty}(-1)^n\sin^2\delta_n(q)\sin n\theta_o\sin n\theta \\
&\qquad - \sum_{n=1}^{\infty}(-1)^n\sin\delta_{\frac{2n-1}{2}}(q)\cos\delta_{\frac{2n-1}{2}}(q)\sin\frac{2n-1}{2}\,\theta_o\sin\frac{2n-1}{2}\,\theta\Bigg\}
\end{aligned}
\tag{3.17}
$$

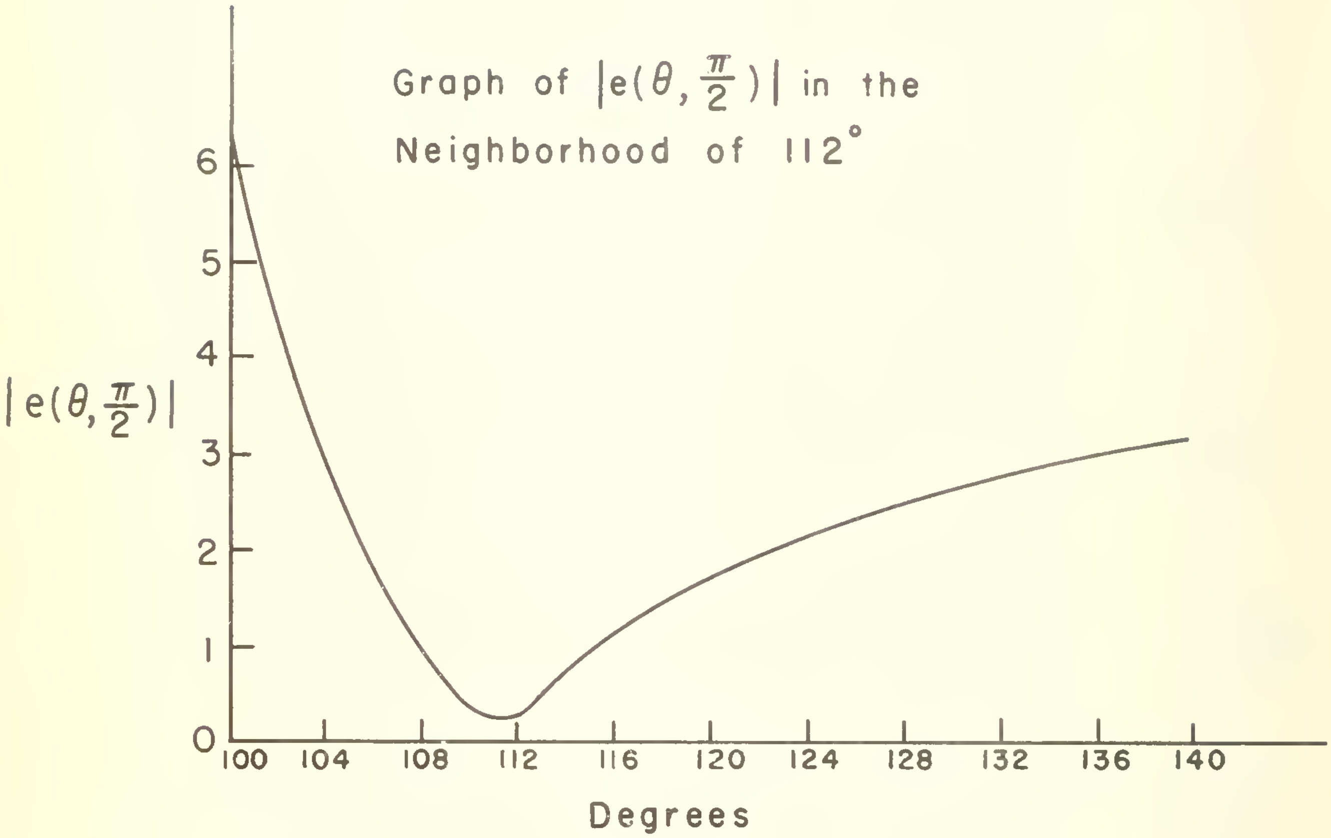
and

$$
\begin{aligned}
\operatorname{Re}\left(\frac{H_1^p}{A(\rho)}\right) &= 4\sin\delta_o'(q)\cos\delta_o'(q) + 8\Bigg\{\sum_{n=1}^{\infty}(-1)^n\sin^2\delta'_{\frac{2n-1}{2}}(q)\cos\frac{2n-1}{2}\cos\frac{2n-1}{2}\,\theta \\
&\qquad + \sum_{n=1}^{\infty}(-1)^n\sin\delta_n'(q)\cos\delta_n'(q)\cos n\theta_o\cos n\theta\Bigg\} \\
\operatorname{Im}\left(\frac{H_1^p}{A(\rho)}\right) &= 4\sin^2\delta_o'(q) + 8\Bigg\{\sum_{n=1}^{\infty}(-1)^n\sin^2\delta_n'(q)\cos n\theta_o\cos n\theta \\
&\qquad - \sum_{n=1}^{\infty}(-1)^n\sin\delta'_{\frac{2n-1}{2}}(q)\cos\delta'_{\frac{2n-1}{2}}(q)\cos\frac{2n-1}{2}\,\theta_o\cos\frac{2n-1}{2}\,\theta\Bigg\}.
\end{aligned}
\tag{3.18}
$$

Graphs of the net scattered amplitudes $|h_1(\theta,\frac{\pi}{2})|$ and $|e_1(\theta,\frac{\pi}{2})|$ are given on pages (16) and (17). The dotted curves represent $|h_1(\theta,\frac{\pi}{2})|$ and $|e_1(\theta,\frac{\pi}{2})|$ when the radius ka of the cylindrical tip is zero and the full curves represent these values for ka = 1. Note that the differences in the pattern of $h_1(\theta,\frac{\pi}{2})$ for ka = 1 and the corresponding one for ka = 0 are greatest in the vicinity of $\theta = 0^o$ and $\theta = 180^o$. In particular, for these angles, $|h(\pi,\frac{\pi}{2})|$ is zero for ka = 0 whereas for ka = 1 it is approximately 2. In the case of $|e_1(\theta,\frac{\pi}{2})|$ it appears that at $\theta = 120^o$, $|e_1(\frac{2}{3}\pi, \frac{\pi}{2})|$ is appreciably less for ka = 1 than it is for ka = 0. The graph on page (17) gives the behavior of $e_1(\theta, \frac{\pi}{2})$ about $\theta = \frac{2}{3}\pi$.

Azimuthal Variation of the Absolute Value of Diffracted Field $|e_1(\theta, \frac{\pi}{2})|$, for Electric Vector Parallel to Half-Plane.





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ield, $\left|h_1(\theta, \frac{\pi}{2})\right|$, for Magnetic
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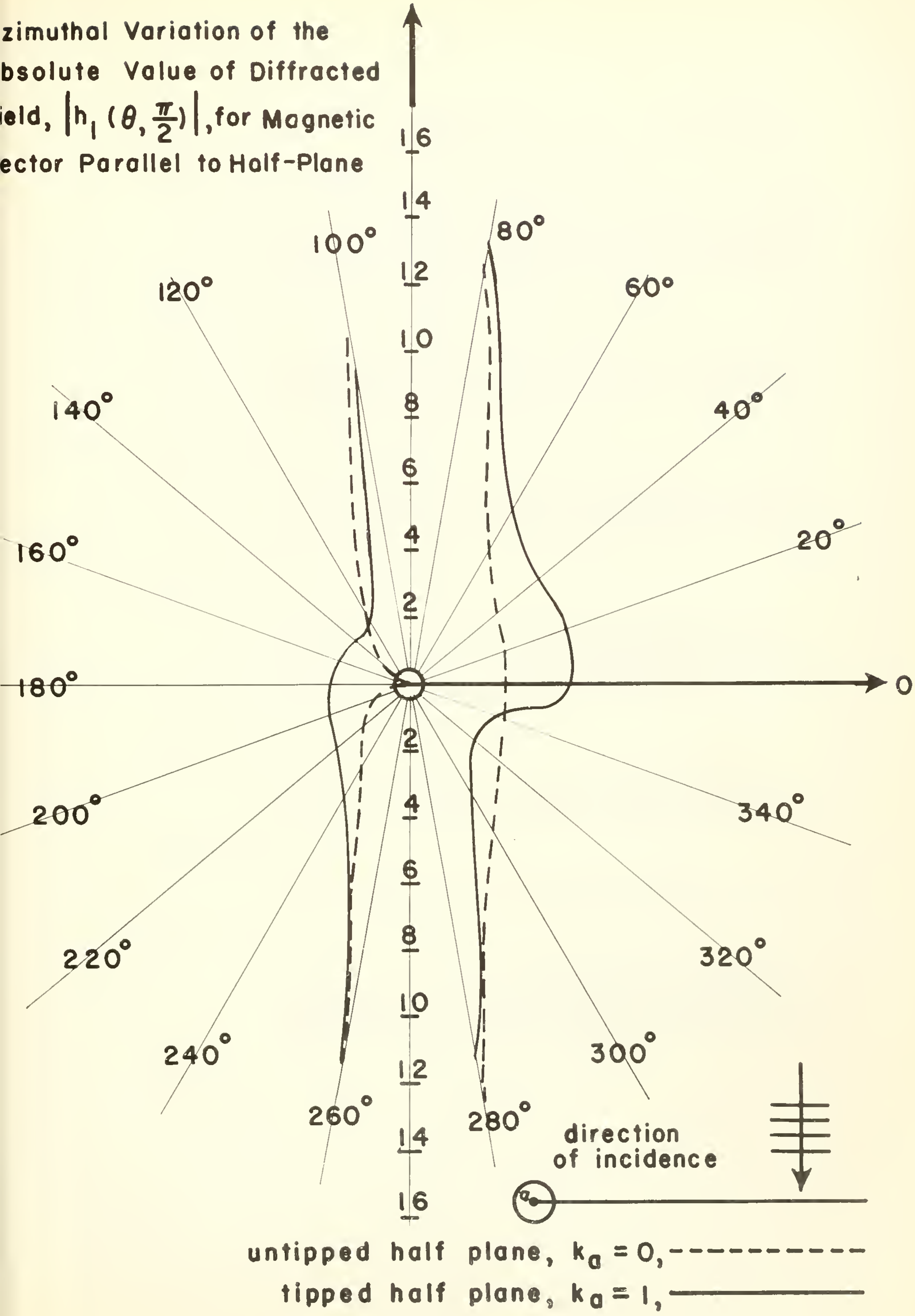

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